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Pj. Ponenti
J. Liandrat

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Institute for Computer Applications in Science and Engineering
NASA Langley Research Center
Hampton, VA 23681-0001

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Hampton, Virginia 23681-0001

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Pj. Ponenti
IMA
Institute for Mathematics and Its Applications
University of Minnesota
206 Church St. S.E., 514 Vincent Hall
Minneapolis, MN

*J. Liandrat**
IRPHE
12 avenue général Leclerc
13003 Marseille, France
and ESM2, Université d'Aix Marseille II, IMT
13451 Marseille, France

Abstract

Wavelet bases are used to generate spaces of approximation for the resolution of bidimensional elliptic and parabolic problems. Under some specific hypotheses relating the properties of the wavelets to the order of the involved operators, it is shown that an approximate solution can be built. This approximation is then stable and converges towards the exact solution. It is designed such that fast algorithms involving biorthogonal multi resolution analyses can be used to resolve the corresponding numerical problems.

Detailed algorithms are provided as well as the results of numerical tests on partial differential equations defined on the bidimensional torus.

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Introduction

Variational approximation methods for partial differential equations are based on weak formulations and on suitable spaces of approximation.

Wavelets are known to be unconditional bases for a large variety of spaces and therefore are good candidates for the generation of approximation spaces for partial differential equation problems. The goal of this paper is to show that moreover, wavelet bases may lead to fast and adaptive numerical resolution of the corresponding approximations.

In this paper, as in previous papers (J. Liandrat and P. Tchamitchian [10], [11]), the wavelets are used to expand the approximated solution of a partial differential equation as well as to approximate the kernel of the differential operator. They are not used only to perform "the linear algebra" (G. Beylkin [2]) related to more classical methods of resolution.

Starting with an expansion of the form $f = \sum_\lambda < f, \Psi_\lambda > \Psi_\lambda$, the solution of the equation $Lu = f$ where L is a constant coefficient elliptic differential operator is $u(x) = \int f(y)K(x,y)dy$ where $K(x,y) = \sum_\lambda L^{-1}\Psi^*_\lambda(x)\Psi(y)$.

Under suitable conditions that will be made precise later, the functions $L^{-1}\Psi^*_\lambda$ as well as $L\Psi_\lambda$ are pseudo wavelets, very close to wavelets (Y. Meyer [13]). This turns out to provide a stable approximation of u. However, the efficiency of the corresponding numerical approximation of u relies, at least in this work, on the hierarchic structure of multiresolution analysis since it provides fast tree algorithms. We will show that, if the operator satisfies suitable conditions that will be made explicit later, then the above mentioned pseudo wavelets are directly related to biorthogonal multiresolution and wavelets. Under these conditions, competitive numerical algorithms involving $O(N)$ or $O(N \log N)$ operations can then be derived.

This paper provides the analysis of the problem and exhibits the corresponding numerical schemes. It is then organized as follows.

The first part is devoted to the general concept of biorthogonal multiresolution analysis on $L^2(\mathbb{R}^n)$. In the second part we focus on the problem of the stability of the multiresolution framework under the action of constant coefficient elliptic operators. The cases of homogeneous and non homogeneous operators are treated separately. The third part deals with the numerical algorithms while the last section is devoted to numerical tests related to the resolution of elliptic and parabolic equations in bidimensional spaces.

I Generalities: Biorthogonal Multi-Resolution Analysis in $L^2(\mathbb{R}^n)$

The concept of multiresolution is at the basis of our construction and we therefore start with a short description of it :

Definition I.1 (Y. Meyer, [12])
A r-regular multiresolution analysis of $L^2(\mathbb{R}^n)$ is a sequence of increasing closed subspaces V_j, $j \in \mathbb{Z}$, $V_j \subset V_{j+1}$, satisfying the following conditions

i) $\bigcap_{-\infty}^{+\infty} V_j = \{0\}$, $\bigcup_{-\infty}^{+\infty} V_j$ is dense in $L^2(\mathbb{R}^n)$;

ii) $f(x) \in V_j \Longleftrightarrow f(2x) \in V_{j+1}$;

iii) $f(x) \in V_0 \Longleftrightarrow f(x-k) \in V_0$, $\forall k \in \mathbb{Z}^n$;

iv) there exists a function Φ in V_0, such that the set of functions $\{\Phi(x-k), k \in \mathbb{Z}^n\}$, is a Riesz basis [†] *for V_0;*

v) the function Φ is regular and localized : Φ is C^{r-1}, $\Phi^{(r-1)}$ is almost everywhere differentiable, and for almost every $x \in \mathbb{R}^n$, for every integer $\alpha \leq r$ and for all integer p, it exists C_p such that

$$|\partial^\alpha \Phi(x)| \leq C_p(1+|x|)^{-p} \ . \tag{1}$$

A consequence of *ii), iii)* and *iv)* is that each V_j is generated by the family of functions $\{\Phi_{jk}(x) = 2^{jn/2}\Phi(2^j x - k),\ k \in \mathbb{Z}^n\}$.

For simplicity reasons, we will only consider the case $n = 2$, but all the results presented to this article can be generalized in any dimension. We will always use for vectors a contracted notation: if e is a bidimensional vector then $e = (e_1, e_2)$.

I.1 Orthogonal multiresolution analysis

To build an orthonormal multiresolution analysis, the Riesz basis $\{\Phi(.-k), k \in \mathbb{Z}^2\}$ is first orthonormalized in such a way that the resulting orthonormal basis is still of the form $\{\Phi(.-k), k \in Z^2\}$.

The wavelets are introduced via the orthogonal complement of V_0 in V_1: W_0.

More precisely, if E is the set of all vertices in the unit cube $[0,1]^2$, and if $E^* = E \setminus \{0\}$, we have the following theorem:

Theorem I.1 *There are 3 functions Ψ^ε, $\varepsilon \in E^*$, in W_0, such that the collection $\{\Psi^\varepsilon(x-k),\ k \in \mathbb{Z}^2, \varepsilon \in E^*\}$ is an orthonormal basis of W_0. Moreover, each Ψ^ε satisfies the same property (1) of regularity and localization as Φ and, moreover, satisfies the following cancelation property*

$$\begin{aligned} \exists\, m \in \mathbb{N} \ , \text{ such that } \forall\ k \ = \ & (k_1,k_2) \in \mathbb{Z}^2,\ 0 \leq k_i \leq m, \\ & \int_{\mathbb{R}^2} x^k \Psi^\varepsilon(x)dx = 0 \ . \end{aligned} \tag{2}$$

[†] A collection of vectors $\{e_\lambda, \lambda \in \Lambda\}$, in a Hilbert space H is a Riesz basis if any vector $x \in H$ can be written in a unique way as $x = \sum a_\lambda\ e_\lambda$ where $(\sum |a_\lambda|^2)^{1/2}$ is finite and defines a norm equivalent to $||x||_H$.

The scaling invariance property *ii)* implies that, for all j, the family $\{\Psi^{\varepsilon}_{jk},\ k \in \mathbb{Z}^2,\ \varepsilon \in E^*\}$ is an orthonormal basis of W_j. We will also use the following contracted notation: $\{\Psi_\lambda,\ \lambda \in \Lambda_j\}$ where $\Lambda_j = \{\lambda = 2^{-j}\left(k + \frac{\varepsilon}{2}\right), k \in \mathbb{Z}^2, \varepsilon \in E^*\}$. Indeed, there is a straightforward bijection between Λ_j and the set of pairs $\{(\varepsilon, k), k \in \mathbb{Z}^2, \varepsilon \in E^*\}$. We will also use the following sets: $\Lambda = \cup_{j \in \mathbb{Z}} \Lambda_j$ and $\Lambda^n_m = \cup^m_{j=n} \Lambda_j$.

From the inclusion $V_0 \subset V_1$ the following *scaling relation* can be derived:

$$\Phi(x) = \sum_{l \in \mathbb{Z}^2} h_l \Phi(2x - l)\ ,$$

while from $W_0 \subset V_1$ one obtains the following *detail relation*:

$$\Psi^{\varepsilon}(x) = \sum_{l \in \mathbb{Z}^2} g^{\varepsilon}_l \Phi(2x - l),\ \forall \varepsilon \in E^*\ .$$

It is very useful to transform these relations using the Fourier transform which is given by the equality

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-i\xi x} dx.$$

Indeed, the scaling relations then become

$$\widehat{\Phi}(\xi) = M_0(\xi/2)\widehat{\Phi}(\xi/2) \tag{3}$$

and

$$\widehat{\Psi}^{\varepsilon}(\xi) = M_{\varepsilon}(\xi/2)\widehat{\Phi}(\xi/2) \tag{4}$$

where $M_0(\xi) = \sum_{l \in \mathbb{Z}^2} h_l e^{-i(\xi.l)}$ and $M_{\varepsilon}(\xi) = \sum_{l \in \mathbb{Z}^2} g^{\varepsilon}_l e^{-i(\xi.l)}$ are C^{∞} 2π periodic functions.

This leads to :

$$\widehat{\Phi}(\xi) = \prod_{j=1}^{\infty} M_0(\frac{\xi}{2^j})\ ,\ \text{and}\ \widehat{\Psi}^{\varepsilon}(\xi) = M_{\varepsilon}(\frac{\xi}{2}) \prod_{j=1}^{\infty} M_0(\frac{\xi}{2^{j+1}}). \tag{5}$$

The following conditions are satisfied

$$\sum_{e \in E} M_{\varepsilon}(\xi + \pi e) M_{\varepsilon'}(\xi + \pi e) = \delta_{\varepsilon\varepsilon'}\ ,\ \forall\ (\varepsilon, \varepsilon') \in E^2 \tag{6}$$

and

$$M_{\varepsilon}(\pi e) = \delta_{\varepsilon_1, e_1}\delta_{\varepsilon_2, e_2},\ \forall (\varepsilon, e) \in E^2 \tag{7}$$

and are called, following electrical engineering terms, the *quadrature mirror filter conditions*. We will also call the functions M_{ε} quadrature mirror filters.

Conversely, it is shown in A. Cohen et al. [4] that four 2π periodic functions $M_\varepsilon(\xi)$, $\varepsilon \in E$ satisfying the quadrature mirror filter conditions (6) generate through (5) an orthogonal multiresolution analysis if some specific conditions are satisfied.

Remark:

- In this paper, we will often use specific multi-resolution analyses of $L^2(\mathbb{R}^2)$ based on a tensorial product of multiresolution analyses of $L^2(\mathbb{R})$. More precisely, such analyses are defined as follows: if $(\mathcal{V}_j)$ is the sequence of spaces of a 1D multiresolution analysis and if $\mathcal{W}_j, \varphi, \psi, m_0$ and m_1 are respectively the related wavelet spaces, the scaling function, the associated wavelet and the quadrature mirror filters, then, the sequence of spaces (V_j), defined as $V_j = \mathcal{V}_j \otimes \mathcal{V}_j$ is a multiresolution analysis in $\mathbb{R}^2$. Moreover, $\Phi(x) = \varphi(x_1)\varphi(x_2)$ is the corresponding scaling function; (W_j), with $W_j = \sum_{\varepsilon \in E^*} W_j^\varepsilon$ are the wavelet spaces; the three generating wavelets are $\Psi_{0,1}(x) = \varphi(x_1)\psi(x_2)$, $\Psi_{1,1}(x) = \psi(x_1)\psi(x_2)$ and $\Psi_{1,0}(x) = \psi(x_1)\varphi(x_2)$; $M_\varepsilon(\xi) = m_{\varepsilon_1}(\xi_1)m_{\varepsilon_2}(\xi_2)$ with $\varepsilon \in E$ are the quadrature mirror filters.

I.2 Biorthogonal Approach

A relaxation of some properties of orthogonal multiresolution analysis can be performed using the one biorthogonal approach. This approach provides some flexibility since it allows to distribute the relevant properties of the multiresolution (number of zero moments, compact support or regularity) to the two involved multiresolution analyses. Moreover, it will turn out to be that the biorthogonal framework is "stable" under the action of a large class of operators while the orthogonal framework is "fragile".

Definition I.2 *We call biorthogonal multiresolution analysis of $L^2(\mathbb{R}^2)$, two multiresolution analysis $(U_j)_{j\in\mathbb{Z}}$ and $(\widetilde{U}_j)_{j\in\mathbb{Z}}$ such that there exists two families of corresponding scaling functions τ and $\widetilde{\tau}$ such that: $\langle \tau_{jk},\ \widetilde{\tau}_{j'k'}\rangle = \delta_{jj'}\delta_{kk'}$ for all $j,\ j' \in \mathbb{Z}$ and, k and $k' \in \mathbb{Z}^2$.*

In this case we define the wavelets spaces X_j and $\widetilde{X}_j$ as $U_{j+1} = U_j \oplus X_j$, $\widetilde{U}_{j+1} = \widetilde{U}_j \oplus \widetilde{X}_j$ with $U_j \perp \widetilde{X}_j$, $\widetilde{U}_j \perp X_j$, and we introduce the functions $\theta_\lambda = 2^j\theta^\varepsilon(2^j. - k)$ and $\widetilde{\theta}_\lambda = 2^j\widetilde{\theta}^\varepsilon(2^j. - k)$, $\varepsilon \in E$, that generate respectively X_j and $\widetilde{X}_j$ and such that $\langle \theta_\lambda,\ \widetilde{\theta}_{\lambda'}\rangle = \delta_{\varepsilon\varepsilon'}\delta_{jj'}\delta_{kk'}$.

Moreover, following the construction of orthogonal multiresolution analysis we define 2×4 filters (i.e, C^∞ 2π periodic functions), P_ε and $\widetilde{P}_\varepsilon$, $\varepsilon \in E$, associated with the two biorthogonal multiresolution analyses. These filters satisfy the biorthogonal quadrature mirror filter relations equivalent to (6) that

are

$$\forall\ \varepsilon\ ,\varepsilon' \text{ and } e' \in E,\ \xi \in [0,\ 2\pi]^2\ ,$$

$$\sum_{e\in E} \widetilde{P}_\varepsilon(\xi+\pi e)\overline{P_{\varepsilon'}(\xi+\pi e)} = \delta_{\varepsilon\varepsilon'}\ , \tag{8}$$

$$P_\varepsilon(\pi e') = \delta_{\varepsilon_1,e_1}\delta_{\varepsilon_2,e_2},\quad \widetilde{P}_\varepsilon(\pi e') = \delta_{\varepsilon_1,e_1}\delta_{\varepsilon_2,e_2}\ .$$

As in the orthonormal case, the generalizations of relations (3) and (4) relate the scaling functions and the wavelets to the filters as

$$\widehat{\tau}(\xi) = P_0(\xi/2)\widehat{\tau}(\xi/2) = \prod_{j=1}^{\infty} P_0(2^{-j}\xi),\ \widehat{\theta^\varepsilon}(\xi) = P_\varepsilon(\xi/2)\widehat{\tau}(\xi/2), \tag{9}$$

$$\widehat{\widetilde{\tau}}(\xi) = \widetilde{P}_0(\xi/2)\widehat{\widetilde{\tau}}(\xi/2) = \prod_{j=1}^{\infty} \widetilde{P}_0(2^{-j}\xi),\ \widehat{\widetilde{\theta}^\varepsilon}(\xi) = \widetilde{P}_\varepsilon(\xi/2)\widehat{\widetilde{\tau}}(\xi/2). \tag{10}$$

As in section I.1, the question to know is under which conditions the biorthogonal filters P_ε and $\widetilde{P}_\varepsilon$ satisfying the quadrature mirror filter conditions (8) define two biorthogonal multiresolution analyses? Again, a specific condition is required and has been formulated in A. Cohen et al. [4]. We will use a weaker version of this formulation adapted to the case of functions with fast decay. It can be expressed in term of the following theorem for which a complete proof can be found in Pj. Ponenti [16] and W. Dahmen and A. Michelli [5].

Theorem I.2 *Let $\alpha > 0$ and let $P_\varepsilon(\xi)$ and $\widetilde{P}_\varepsilon(\xi)$, $\varepsilon \in E$, be eight C^α 2π periodic functions satisfying the biorthogonal quadrature mirror filters conditions (8). Defining τ, $\widetilde{\tau}$, θ, and $\widetilde{\theta}$ using formula (9) and (10), if*

- *there exist C and $\mu > 0$ such that for all $\xi \in \mathbb{R}^2$*

$$|\widehat{\tau}(\xi)| \leq C(1+|\xi|)^{-1-\mu},\qquad |\widehat{\widetilde{\tau}}(\xi)| \leq C(1+|\xi|)^{-1-\mu}\ , \tag{11}$$

- *$\forall\ k\ ,$ et $k' \in \mathbb{Z}^2$,*

$$\langle \tau(x-k), \widetilde{\tau}(x-k')\rangle = \delta_{kk'}, \tag{12}$$

- *and if*

$$\int_{\mathbb{R}^2} \frac{|\widehat{\theta}(\xi)|^2}{|\xi|^2} d\xi < \infty,\qquad \int_{\mathbb{R}^2} \frac{|\widehat{\widetilde{\theta}}(\xi)|^2}{|\xi|^2} d\xi < \infty\ , \tag{13}$$

then

- *the sequences of subspaces $(U_j)_{j\in\mathbb{Z}}$, and $(\widetilde{U}_j)_{j\in\mathbb{Z}}$ generated respectively by $\{\tau_{jk}, k \in \mathbb{Z}^2\}$ and $\{\widetilde{\tau}_{jk}, k \in \mathbb{Z}^2\}$ are two biorthogonal multiresolution analyses;*
- *the wavelet families $\{\ \theta_\lambda(x) = 2^j\Psi^\varepsilon(2^j x - k),\ \lambda \in \Lambda\ \}$ and $\{\widetilde{\theta}_\lambda(x) = 2^j\widetilde{\theta}^\varepsilon(2^j x - k),\ \lambda \in \Lambda\}$ are two biorthogonal Riesz bases of $L^2(\mathbb{R}^2)$.*

II Constant Coefficient Elliptic Differential Operators and Biorthogonal Multiresolution Analysis

II.1 General results

The starting point of this section is the following remark. Given (Ψ_λ) a family of orthonormal wavelets and knowing $f = \sum \langle f, \Psi_\lambda \rangle \Psi_\lambda$, the solution of the equation

$$Lu = f \ , \tag{14}$$

where L is an elliptic operator of order s is, at least formally,

$$u = \sum \langle f, \Psi_\lambda \rangle L^{-1}[\Psi_\lambda] \ . \tag{15}$$

When L and L^{-1} are bounded on $L^2(\mathbb{R}^2)$, the families $\{L^{-1}\Psi_\lambda\}$ and $\{L^*\Psi_\lambda\}$ are two biorthogonal Riesz bases.

The question we address now is related to what happens in the specific case of wavelets when the operator L is unbounded (as in the case of a differential operator).

In the following paragraphs, we first see that, assuming some compatibility conditions between Ψ^ϵ and L, the two families of functions $\{L^{-1}\Psi_\lambda\}$ and $\{L^*\Psi_\lambda\}$ are wavelets or pseudo wavelets (Y. Meyer [13]) depending on whether the operator is homogeneous or not. Then, we show that in some cases, a biorthogonal framework embedding $\{L^{-1}\Psi_\lambda\}$ and $\{L^*\Psi_\lambda\}$ can be built.

To be more precise, we take (V_j) an r-regular multiresolution of $L^2(\mathbb{R}^2)$ constructed using a tensorial product of two 1D multiresolutions and L a constant coefficient elliptic differential operator. Let us write $L = \sum_{0 \le \alpha \le s} a_\alpha D^\alpha$ where D is the operator $\frac{1}{i}\frac{\partial}{\partial x}$. We define in a standard way the symbol of L as

$$\sigma(\xi) = \sum a_\alpha \xi^\alpha. \tag{16}$$

It is a polynomial in ξ of degree s and, if $f \in C^\infty(\mathbb{R}^2)$, we have the well known formula

$$\forall \xi \in \mathbb{R}^2 \ , \hat{Lf}(\xi) = \hat{f}(\xi)\sigma(\xi). \tag{17}$$

We note formally

$$\theta^\epsilon = L^*\Psi^\epsilon \text{ and } \widetilde{\theta}^\epsilon = L^{-1}\Psi^\epsilon \tag{18}$$

and more generally,

$$\theta^\epsilon_\lambda = 2^{-js} L^*\Psi^\epsilon_\lambda \text{ and } \widetilde{\theta}^\epsilon_\lambda = 2^{js} L^{-1}\Psi^\epsilon_\lambda. \tag{19}$$

Note that when L is homogeneous, $\theta^\epsilon_\lambda(x) = 2^j \theta^\epsilon(2^j x - k)$ and $\widetilde{\theta}^\epsilon_\lambda(x) = 2^j \widetilde{\theta}^\epsilon(2^j x - k)$ while, in general this is not true.

Then we have

Theorem II.1 *Given a family of r regular wavelets with $m+1$ zero moments, if L is a homogeneous operator or if L is an inhomogeneous operator with a strictly positive symbol of order $s > 0$, and such that $r \geq s$ and $m \geq s$, then*

Regularity: $\theta^\varepsilon \in H^{r-s}$ and $\widetilde{\theta}^\varepsilon \in H^{r+s}$

Localization: for all multi-indices[‡] γ' and γ such that $|\gamma'| \leq r-s$ and $|\gamma| \leq r+s-1$ and all integers $l \in \mathbb{N}$,

$$\begin{cases} |\partial^{\gamma'}\theta^\varepsilon_\lambda(x)| & \leq & C_{\gamma'}\ 2^{j|\gamma'|}2^j & \left(1+2^j|x-\lambda|\right)^{-l} \\ |\partial^{\gamma}\widetilde{\theta}^\varepsilon_\lambda(x)| & \leq & C_{\gamma}\ 2^{j|\gamma|}2^j & \left(1+2^j|x-\lambda|\right)^{-3+s-m-|\gamma|} \end{cases}$$

Cancelation: Let $x^k = (x_1^{k_1},\ x_2^{k_2})$,

$$\begin{cases} \displaystyle\int_{\mathbb{R}^2} x^k\theta^\varepsilon(x)dx &=& 0, & 0 \leq |k| \leq m+a \\ \displaystyle\int_{\mathbb{R}^2} x^k\widetilde{\theta}^\varepsilon(x)dx &=& 0, & 0 \leq |k| \leq m-a\ , \end{cases}$$

where $a = s$ in the homogeneous case and $a = 0$ in the inhomogeneous one.

A complete proof of this theorem can be found in Y. Meyer [13] and Ph. Tchamitchian [17].

Remark:

- Following Y. Meyer [13] and according to Theorem II.1, a factorization of the operators L^* and L^{-1} can be performed as $L^* = C \circ \Gamma^{-s}$ and $L^{-1} = \widetilde{C} \circ \Gamma^s$ where Γ is a diagonal operator in the wavelet basis defined as: $\psi_\lambda \mapsto 2^j\psi_\lambda$ and where C and $\widetilde{C}$ are bounded on L^2 and defined by

$$C\ :\ \psi_\lambda \mapsto \theta_\lambda\ , \qquad \widetilde{C}\ :\ \psi_\lambda \mapsto \widetilde{\theta}_\lambda\ . \tag{20}$$

  The operators C and $\widetilde{C}$ act just as a transformation between two bases. The operator Γ is nothing else but the classical preconditioning operator for elliptic problems (S. Jaffard [8]) that mimics a diagonal derivation in the wavelet basis.

  In other words, thanks to this factorization, the computation of the image of a function by an elliptic operator or its inverse can be transformed into a well conditioned problem using a diagonal operator in a suitable wavelet basis. This is essential since it provides the numerical stability of the further developed algorithm.

We can then rewrite (15) as

$$u = \sum 2^{-js}\langle f,\ \Psi_\lambda\rangle\widetilde{\theta}_\lambda\ . \tag{21}$$

[‡]We call a multi-index any couple of integer $\gamma' = (\gamma_1, \gamma_2)$and $\partial^{\gamma'}\theta = \frac{\partial^{\gamma_1}}{\partial x_1^{\gamma_1}}\frac{\partial^{\gamma_2}}{\partial x_2^{\gamma_2}}$

An important issue, as far as numerical applications are concerned, is the computation of the sum (21). Indeed, even if this sum corresponds to a pseudo wavelet decomposition, fast algorithms for the computation of the sum are not available. In the framework of this paper, the fast algorithms are linked to the concept of multiresolution analysis presented in section I. We prove in the following sections that, under suitable conditions, the construction of a multiresolution analysis embedding the function $\widetilde{\theta}_\lambda$ is possible. Moreover, we provide explicit expressions of the quadrature mirror filters required for the fast implementation of (21).

The starting point of our construction is due to P.G. Lemarié [9] who constructed in the one dimensional case two biorthogonal multiresolution analyses from an original orthogonal one and from the derivata operator. We generalize this approach to any dimensions and for any homogeneous elliptic operator.

In contrast to the classical constructions of multiresolution analysis, this is an inverse problem. Indeed, knowing the two dual wavelet bases we can define two sequences of subspaces (X_j) and $(\widetilde{X}_j)$:

$$\begin{aligned} X_j &= \text{span}\{\theta_\lambda, \lambda \in \Lambda_l, l < j\} \\ \widetilde{X}_j &= \text{span}\{\widetilde{\theta}_\lambda, \lambda \in \Lambda_l, l < j\} \ , \end{aligned} \tag{22}$$

The open question is the following: how can we construct two sequences of subspaces (U_j) and $(\widetilde{U}_j)$ for which (X_j) and $(\widetilde{X}_j)$ play the role of two biorthogonal wavelet spaces?

In other words, the problem is the construction of the generalized scaling functions τ and $\widetilde{\tau}$ related to θ and $\widetilde{\theta}$.

In the case of non homogeneous elliptic operators the approach used in the homogeneous case can not be transposed and we will not be able to define a multiresolution framework embedding the space sequences (U_j) and $(\widetilde{U}_j)$. However, we will show that the essential property of embedding spaces as well as the existence of scaling (3) and detail (4) relations can be saved. This will allow us to derive fast and stable algorithms to sum up (21) even in the case of non homogeneous operators.

II.2 The case of homogeneous elliptic operator

In that case the natural candidates for τ: $L^{-1}\Phi_\lambda$, are not defined in L^2 for the basic reason that Φ does not belong to the range of L, or, in other words, suffers from a lack of zero moments. Using a preconditioning operator, we will adapt the function Φ to the operator L^{-1} (i.e. we will transform Φ so that it enters the range of L) while preserving the two scale relations (3) and (4). We will then check that the resulting multiresolution analysis is fitted to the functions θ_λ and $\widetilde{\theta}_\lambda$ previously defined.

More precisely, we have the following theorem:

Theorem II.2 *Let (V_j) be the family of embedded spaces of an r-regular multiresolution analysis of $L^2(\mathbb{R}^2)$, let L be a homogeneous operator of order s and of symbol σ and let S be a 2π periodic function, not vanishing on $]0,2\pi[^2$ and equivalent to σ in zero. If $m > s+1$, $r > s+1$, and if the eight functions $P_\varepsilon(\xi)$ and $\widetilde{P}_\varepsilon(\xi)$, $\varepsilon \in E$ defined as*

$$P_0(\xi) = 2^s \frac{\overline{S(\xi)}}{\overline{S(2\xi)}} M_0(\xi), \quad \widetilde{P}_0(\xi) = \frac{1}{2^s} \frac{S(2\xi)}{S(\xi)} M_0(\xi) , \tag{23}$$

$$P_\varepsilon(\xi) = 2^s \overline{S(\xi)}\, M_\varepsilon(\xi), \quad \widetilde{P}_\varepsilon(\xi) = \frac{1}{2^s} \frac{M_\varepsilon(\xi)}{S(\xi)} \tag{24}$$

are C^α, $\alpha > 0$ then they satisfy the quadrature mirror filter conditions (8) and define two biorthogonal multiresolution analyses. The corresponding wavelets are the functions θ and $\widetilde{\theta}$ and the scaling functions τ and $\widetilde{\tau}$ are derived following (9) and (10).

Proof:

□ By construction all the filters are C^α with some $\alpha > 0$ and they satisfy the biorthogonal quadrature mirror filter conditions (8).

The only point to prove is the convergence in $L^2(\mathbb{R}^2)$ of the infinite products (9) and (10) defining the two scaling functions. We will use the following lemma:

Lemma II.1 *Let $p(x)$, $x \in \mathbb{R}^2$, be a homogeneous polynomial of degree s, and let S and C, be 2π periodic functions; then the following propositions are equivalent:*

i)

$$\prod_{j=1}^{\infty} C(2^{-j}x) = \frac{S(x)}{p(x)} , \tag{25}$$

ii)

$$S(x) = 2^s C(x/2) S(x/2) , \tag{26}$$

$$S(x) \underset{x\to 0}{\sim} p(x) . \tag{27}$$

Proof:

□ The equation (26) is obtained from (25) written for x and $x/2$, while (27) is derived from (25) when $x \to 0$ since necessarily $C(0) = 1$.

Conversely, (25) is obtained from (26) . Indeed, since

$$\frac{S(x)}{p(x)} = C(x/2)\frac{S(x/2)}{p(x/2)} = \frac{S(2^{-N}x)}{p(2^{-N}x)} \prod_{j=1}^{N} C(2^{-j}x) ,$$

thanks to (27) we obtain (25) when $N \to \infty$. ■

This lemma allows us to calculate the infinite product (9) and (10), and to get:

$$\begin{aligned}
\widehat{\tau}(\xi) &= \prod_{j=1}^{\infty} P_0(2^{-j}\xi) = \prod_{j=1}^{\infty} \frac{1}{\overline{C}(2^{-j}\xi)} \prod_{j=1}^{\infty} m_0(2^{-j}\xi_1) m_0(2^{-j}\xi_2) \\
\widehat{\widetilde{\tau}}(\xi) &= \prod_{j=1}^{\infty} \widetilde{P}_0(2^{-j}\xi) = \prod_{j=1}^{\infty} C(2^{-j}\xi) \prod_{j=1}^{\infty} m_0(2^{-j}\xi_1) m_0(2^{-j}\xi_2) ,
\end{aligned}$$

and finally,

$$\widehat{\widetilde{\tau}}(\xi) = \frac{S(\xi)}{\sigma(\xi)} \widehat{\Phi}(\xi) , \ \widehat{\tau}(\xi) = \frac{\overline{\sigma}(\xi)}{\overline{S}(\xi)} \widehat{\Phi}(\xi) . \tag{28}$$

The function Φ being r-regular the conditions (11) are trivially satisfied. Defining the function θ et $\widetilde{\theta}$ by (9) and (10) we get

$$\begin{aligned}
\widehat{\theta^{\varepsilon}}(\xi) &= P_{\varepsilon}(\xi/2)\widehat{\tau}(\xi/2) = \overline{\sigma}(\xi)\widetilde{\Psi}(\xi) \\
\widehat{\widetilde{\theta}}^{\varepsilon}(\xi) &= \widetilde{P}_{\varepsilon}(\xi/2)\widehat{\widetilde{\tau}}(\xi/2) = \frac{\widetilde{\Psi}(\xi)}{\sigma(\xi)} .
\end{aligned} \tag{29}$$

The wavelet admissible condition (13) is immediately satisfied thanks to Theorem II.1. Finally the assumption (12) is satisfied by construction, that completes the proof. ■

Remarks:

- Note that here, thanks to homogeneity, the subscript λ recovers its classical "wavelet" meaning since in that case, $\theta_\lambda(x) = 2^{j/2}\theta(2^j x - k)$ and $\widetilde{\theta}_\lambda(x) = 2^{j/2}\widetilde{\theta}((2^j x - k)$.

- The relation (25) is a generalization in any dimension, of the classical formula

$$\prod_{j=1}^{\infty} \cos(2^{-j}x) = \frac{\sin(x)}{x}$$

used by P.G. Lemarie for the first order differential operator in one dimension.

- The function S can be interpreted as the symbol of a difference operator that we will call D_L. If S is a trigonometric polynomial, then D_L is a finite-difference operator and S is C^∞. The fact that $S(x) \underset{x\to 0}{\sim} \sigma(x)$ is just the translation that D_L is consistent with L. Moreover, S removes exactly the singularity of $\widetilde{\tau}$ for $\xi = 0$. Conversely for τ, the singularity

given by S at points $(2\pi n, 2\pi n)$, $n \in \mathbb{Z}$, will be removed exactly by σ in zero and by the zeros of $\widehat{\Phi}$ at points $(2\pi n, 2\pi n)$, $n \in \mathbb{Z}^*$. Notice that this last point won't be true for inhomogeneous operators.

From a certain point of view, D_L can be seen as a preconditioner for L since τ and $\tilde{\tau}$ are defined by

$$\tau = L^{-1} D_L \Phi \text{ and } \tilde{\tau} = (D_L^*)^{-1} L^* \Phi. \tag{30}$$

- In one dimension there is a canonical choice for S and therefore for D_L such that, if the function Φ and Ψ have a compact support, then τ, $\widetilde{\tau}$, θ, and $\widetilde{\theta}$ are also compactly supported. Indeed, in that case we have necessarily $\sigma(\xi) = a\xi^s, a \in \mathbb{C}$. Therefore, the canonical choice for S is $S(\xi) = a(-i)^s \left(1 - e^{-i\xi}\right)^s$ and D_L is then a non-centered finite-difference approximation of L of order 1. Indeed, it is well known (see I. Daubechies [6]) that the quadrature mirror filters related to orthogonal compactly supported functions Φ and Ψ are: $m_0(\xi) = \left(1 + e^{i\xi}\right)^m \mathcal{L}(\xi)$ and $m_1(\xi) = e^{i\xi} \left(1 - e^{-i\xi}\right)^m \overline{\mathcal{L}(\xi + \pi)}$ where $\mathcal{L}$ is a finite trigonometric polynomial. Then we get

$$\begin{aligned}
P_0(\xi) &= 2^s \left(1 + e^{i\xi}\right)^{m-s} \mathcal{L}(\xi), \\
P_1(\xi) &= 2^s \overline{a} i^s e^{i\xi} \left(1 - e^{-i\xi}\right)^m \left(1 - e^{i\xi}\right)^s \overline{\mathcal{L}(\xi + \pi)}, \\
\widetilde{P}_0(\xi) &= 2^{-s} \left(1 + e^{i\xi}\right)^m \left(1 + e^{-i\xi}\right)^s \mathcal{L}(\xi), \\
\widetilde{P}_1(\xi) &= 2^{-s} a^{-1} i^s e^{i\xi} \left(1 - e^{-i\xi}\right)^{m-s} \overline{\mathcal{L}(\xi + \pi)},
\end{aligned}$$

which proves that P_0, P_1, $\widetilde{P}_0$ and $\widetilde{P}_1$ are also finite trigonometric polynomials. Then, using the following lemma borrowed from G. Deslauriers and S. Dubuc [7] we deduce that the functions $\tilde{\tau}$, τ, $\widetilde{\theta}$, and θ have compact support.

Lemma II.2

If $\Gamma(\xi) = \sum_{n=N_1}^{N_2} \gamma_n e^{-in\xi}$ *with* $\sum_{n=N_1}^{N_2} \gamma_n = 1$, *then* $\prod_{j=1}^{\infty} \Gamma(2^{-j}\xi)$ *is an entire function of exponential type. In particular, it is the Fourier transform of a distribution with support in* $[N_1, N_2]$.

Clearly, this canonical form is no longer available if the space dimension is larger than 1 since the multidimensional quadrature mirror filters can not be factorized as above.

II.3 The case of inhomogeneous elliptic operator

Here the non homogeneous property of the operator is obviously not adapted to the scale invariance property of the multiresolution analysis. We will see

however, that introducing at each level a new scaling function, an embedded family of spaces can be constructed which preserves the mathematical properties relevant for numerical applications.

The natural candidates for τ_λ are a the functions $L^{-1}\Phi_\lambda$. They are well defined in $L^2(\mathbb{R}^2)$ but suffer now from a lack of localization when j increases. Indeed, we have

$$\lim_{\substack{j \longrightarrow +\infty \\ k2^{-j} \longrightarrow x_0}} \| \ 2^{-jn/2} L^{-1} \left[\varphi_{jk}\right](x) - G(x - 2^{-j}k) \ \| = 0$$

where G is the Greens function of the operator L defined as $\hat{G}(\xi) = 1/\sigma(\xi)$. For example, when $L = 1-\Delta$, $\hat{G}(\xi) = 1/(1+\xi^2)$ and $G(x) = e^{-|x|}$. G decreases fast but mathematical and numerical difficulties come from the fact that the family of functions $\{G(x - k2^{-j}), k \in \mathbb{Z}^2\}$ is not a set of functions rescaled with j (in other words, this family is not obtained by rescaling and translation of a single initial function). This implies that the control of the localization by the index j is lost. It follows that the family $\{L^{-1}\Phi_\lambda, \ k \in \mathbb{Z}^2\}$ is not a good basis to reconstruct our solution.

Let us show now that, however, a process very close to the one used in the homogeneous case will provide an efficient algorithm for the summation of formula (21).

We mimic the construction performed in the homogeneous case. Let (V_j) be an r-regular multiresolution analysis, let L be an elliptic operator of order s with constant coefficients and σ its symbol (we now suppose $\sigma(\xi) \geq \sigma_0 > 0 \ \ \forall \xi$). Let us also define the homogeneous polynomial of order s, $\dot{\sigma}$, as the principal part of σ, and let $S(\xi)$ be a C^∞ 2π periodic function with $S(\xi) \underset{\xi \to 0}{\sim} \xi^n$, where n will be fixed later.

Then, $\forall j \in \mathbb{Z}$, we define a difference operator D_j by its symbol $S(\xi/2^j)$.

Following the previous section, we define $\forall j, k \in \mathbb{Z}$

$$\widetilde{\tau}_{jk} \ = \ 2^{js} L^{-1} D_j \Phi_{jk} \tag{31}$$

and

$$\widetilde{\theta}_{jk} \ = \ 2^{js} L^{-1} (\Psi_{jk}) . \tag{32}$$

By construction, and thanks to the fact that L is a constant coefficient operator we have $\widetilde{\tau}_{jk}(x) = \widetilde{\tau}_j(x - \frac{k}{2^j})$ and $\widetilde{\theta}_{jk}(x) = \widetilde{\theta}_j(x - \frac{k}{2^j})$ where

$$\widehat{\widetilde{\tau}}_j(\xi) = 2^{j(s-1)} S(\xi/2^j) \frac{\widehat{\Phi}(\xi/2^j)}{\sigma(\xi)} \ , \tag{33}$$

and

$$\widehat{\widetilde{\theta}}_j^{\varepsilon}(\xi) = 2^{j(s-1)} \frac{\widehat{\Psi}(\xi/2^j)}{\sigma(\xi)} . \tag{34}$$

Remark

- The functions $\tilde{\tau}_j$ mimic the function $\tilde{\tau}$ defined in (28). Unfortunately, it is not possible to define the equivalent of τ (28) since $D_j^{-1} L\Phi_{jk} \notin L^2$. Note however that, by chance, (15) involves directly the $\tilde{\theta}_\lambda$ functions.

Then, with $\widetilde{P}_0$ and $\widetilde{P}_\varepsilon$ defined in (23) and 24 we get the following scaling and detail relations:

$$\begin{aligned} \widehat{\tilde{\tau}}_j(\xi) &= 2\widetilde{P}_0(\xi/2^{j+1})\widehat{\tilde{\tau}}_{j+1}(\xi) \\ \widehat{\tilde{\theta}}_j^\varepsilon(\xi) &= 2\widetilde{P}_\varepsilon(\xi/2^{j+1})\widehat{\tilde{\theta}}_{j+1}^\varepsilon(\xi)\ ,\ \forall \varepsilon \in E^* . \end{aligned} \tag{35}$$

Remark:

- An important point is that the filters P_ε are independent of the scale index j as it is originally the case for standard multiresolution analysis. Furthermore , since they are defined by (23) and 24, the filters P_ε are directly related to the homogeneous operator $\dot{L}$ of symbol $\dot{\sigma}$ if $n = s$. This point is essential since it means that if D_0 is consistent to L, then the tree algorithms related to the multiresolution spaces $(\widetilde{U}_j)$ and used to sum up (21) are stable even if the functions $\tilde{\tau}_j(x)$ are not standard scaling functions. Indeed, $S(\xi)/\sigma(\xi) \rightarrow_{\xi \rightarrow 0} 0$ and therefore $\widehat{\tilde{\tau}}_j(0) = 0$.

In other words, even if the functions $\tilde{\tau}_{jk}(x)$ are used as scaling functions on the range of L, they have zero moments as wavelets have.

Finally, we can prove the following theorem:

Theorem II.3 *For $0 < n \leq s$, $s \leq r$ and $s \leq m$, the functions $\tilde{\tau}_{jk}$ defined by (33) and (31) satisfy*

$$|\partial^\gamma \tilde{\tau}_{jk}(x)| \leq C'_\gamma\ 2^{j(|\gamma|-s)} 2^j \left(1 + 2^j |x - 2^{-j}k|\right)^{-n+s-m-3-|\gamma|} . \tag{36}$$

If $n = s$ and if $\widetilde{P}_\varepsilon(\xi)$, $\varepsilon \in E$ are C^α, then stable tree algorithms are available in $\bigoplus_{\ell=0}^{j-1} X_\ell$, ie,

$\exists 0 < C \leq C' < +\infty$ such that if $f = \sum_{\lambda \in \Lambda_0^{j-1}} d_\lambda \tilde{\theta}_\lambda$, then

$$C \sum_\lambda |d_\lambda|^2 \leq \| f \|^2 \leq C' \sum_\lambda |d_\lambda|^2 . \tag{37}$$

Proof:

- Since $S \in C^\infty$ the we can apply Theorem II.1 that proves the localization inequality (36).

- The relations (35) defines directly the tree algorithms required to compute the coefficients $\{c_{jk}\}$ such that

$$\sum_{\lambda \in \Lambda_0^{j-1}} d_\lambda \widetilde{\theta}_\lambda = \sum_k c_{jk} \widetilde{\tau}_{jk}$$

and since, when $n = s$, the involved quadrature mirror filters could be related to the homogeneous operator $\dot{L}$, the stability of the algorithm is equivalent to the stability of the transform $\{c_{jk}\} \mapsto f$ for $f \in \bigoplus_{\ell=0}^{j-1} X_\ell$.

- This transform is stable if and only if the family $\{\widetilde{\tau}_{jk},\ k \in \mathbb{Z}^2\}$ is a Riesz basis of $\bigoplus_{\ell=0}^{j-1} \widetilde{X}_\ell$.

We have

$$\begin{aligned} \| f \|^2 &= \int \left| \sum_{k \in \mathbb{Z}^2} c_{jk}\ \widetilde{\tau}_{jk} \right|^2 dx \\ &= \int_{[0,\,2\pi]} \left| \sum_{k \in \mathbb{Z}^2} c_{jk}\ e^{-i2^{-j}(k.\xi)} \right|^2 \left| S(\xi/2^j) \right|^2 \times \\ &\quad \sum_l \left| \frac{\widehat{\Phi}(2^{-j}\xi + 2l\pi)}{\sigma(\xi + 2l\pi)} \right|^2 d\xi \ . \end{aligned}$$

Since $\{\Phi_{jk}\}$ is a Riesz basis, since $S(\xi)$ is bounded, and since σ is bounded from below, then $\| f \|^2 \leq C' \sum |c_{jk}|^2$, which is the second part of the inequality (37).

To prove the first part of (37) we use again the fact that the filters $\widetilde{P}_0$ and $\widetilde{P}_\varepsilon$ are related to $\dot{L}$.

Indeed, if we define $\dot{\widetilde{\theta}}_\lambda$ replacing σ by $\dot{\sigma}$ in (29) and if we define $\dot{f}$ as $\dot{f} = \sum_\lambda d_\lambda \dot{\widetilde{\theta}}_\lambda$ then the transform $\dot{f} \mapsto \{d_\lambda\}$ is stable. Moreover thanks to theorem II.1, the operator $f \mapsto \dot{f}$ which can be also defined as $\Psi_\lambda \mapsto \dot{\widetilde{\theta}}_\lambda$ is also bounded (Y. Meyer [13]). Therefore the operator $f \mapsto \{d_\lambda\}$ is bounded that is the first part of (37) and that completes the proof.

■

III Approximation and Numerical Resolution of Elliptic Problem on the Torus

This section is devoted to the approximation of elliptic problems on a sequence of embedded Galerkin spaces associated with a multiresolution analysis and to the corresponding numerical algorithms.

Classically, we will consider the problem with periodic boundary conditions to avoid the difficulties of general boundary conditions. We will use a r-regular multiresolution analysis of the torus $T_{[0,1]^2} = (\mathbb{R}/\mathbb{Z})^2$ constructed using a classical periodization technique (Y. Meyer [12]). We take as granted that, with minor modifications, the results proved on the whole line can be transposed to the torus. In that section, homogeneous and inhomogeneous operators will be treated similarly.

III.1 General formulation

The general formulation of the problem is

$$(\mathcal{P}) \quad \begin{cases} \textit{Find } u \in T_{[0,1]^2} \textit{ such that} \\ \qquad Lu = f \qquad (38) \\ \textit{with } f \in T_{[0,1]^2} \textit{ and } L \textit{ a constant coefficient elliptic operator of order } s. \end{cases}$$

Standard variational approximation (P.A. Raviart and J.M. Thomas [14]) leads us to look for the solution of a weak problem in so called Galerkin approximation spaces V_ε, where ε is a scale related to V_ε with $V_\varepsilon \underset{\varepsilon \to 0}{\to} T_{[0,1]^2}$.

A natural choice for V_ε is $V_\varepsilon = V_j^{\mathcal{P}}$ [§], where $V_j^{\mathcal{P}}$ belongs to a multiresolution analysis of $T_{[0,1]^2}$ of the type described above. Indeed, we then have the following inequality guaranteed if the involved multiresolution analysis is r-regular,

$$\forall\, s < r,\ \exists\, c > 0\ \forall f \in H^s\ \ ||f - \Pi_{V_j^{\mathcal{P}}} f||_2 \ \leq\ C\, 2^{-js}\ ||f||_{H^s} \qquad (39)$$

where $\Pi_{V_j^{\mathcal{P}}}$, $j \leq 0$, stands for the orthogonal projection on $V_j^{\mathcal{P}}$.

Then a standard Galerkin approximation writes

$$(\mathcal{P}') \quad \begin{cases} \textit{Find } u_p \in V_p^{\mathcal{P}} \subset T_{[0,1]^2} \textit{ such that} \\ \qquad \Pi_{V_p^{\mathcal{P}}} L \widetilde{\Pi}_{V_p^{\mathcal{P}}} u_p \ = \ \Pi_{V_p^{\mathcal{P}}} f. \qquad (40) \end{cases}$$

where $\widetilde{\Pi}_{V_j^{\mathcal{P}}}$, $j \leq 0$, stands for the extension operator from $V_j^{\mathcal{P}}$ to $T_{[0,1]^2}$.

This approach leads us to replace L by the approximation $\Pi_{V_p^{\mathcal{P}}} L \widetilde{\Pi}_{V_p^{\mathcal{P}}}$. The corresponding numerical algorithms are reduced to linear system solvers once a basis of $V_p^{\mathcal{P}}$ has been chosen (P.A. Raviart et al. [14]).

[§] The symbol $.^{\mathcal{P}}$ stands for the periodization operator on $[0,\ 1]^2$. We recall that $\dim V_j^{\mathcal{P}} = 2^{2j} = 1/3\ \dim W_j$ and that $V_0^{\mathcal{P}} = \text{span}\{\Phi_{00} = 1\}$.

Numerically, an optimal choice for the expansion basis is the wavelet basis of $V_p^{\mathcal{P}}$: $\{\Phi_{00}^{\mathcal{P}}, \Psi_\lambda^{\mathcal{P}}, \lambda \in \Lambda_0^{p-1}\}$¶, since the corresponding stiffness matrix, is sparse and can be easily (i.e using diagonal matrices) uniformally preconditioned (S. Jaffard [8]).

III.2 A different approximation

The main purpose of this paper is to define a different approximation of L and the corresponding numerical algorithm. Taking the set of functions $\{\widetilde{\theta}_\lambda^{\mathcal{P}},\ \lambda \in \Lambda_0^{p-1}\}$ (defined in (31)) and defining $\widetilde{U}_p^{\mathcal{P}} = \text{span}\{L^{-1}\Phi_{00}^{\mathcal{P}},\ \widetilde{\theta}_\lambda^{\mathcal{P}},\ \lambda \in \Lambda_p\}$, we get an approximation of u as

$$u_p = \sum_{\lambda \in \Lambda_0^p} 2^{-js} \langle f, \Psi_\lambda^{\mathcal{P}} \rangle \widetilde{\theta}_\lambda^{\mathcal{P}} + \langle f, \Phi_{00}^{\mathcal{P}} \rangle L^{-1}\Phi_{00}^{\mathcal{P}} \,. \tag{41}$$

Indeed,

$$u_p = L^{-1}\Pi_{V_p^{\mathcal{P}}} f = P_{\widetilde{U}_p^{\mathcal{P}}} L^{-1} f \tag{42}$$

where $P_{\widetilde{U}_p^{\mathcal{P}}}$ is the projection on $\widetilde{U}_p^{\mathcal{P}}$ orthogonal to $U_p^{\mathcal{P}}$.

This formulation of u_p shows that the convergence of u_p towards u when $p \rightarrow \infty$ is straightforward since the set $\widetilde{U}_p^{\mathcal{P}}$ is a family of Galerkin spaces for the suitable space of definition of u.

Moreover, the stability of the algorithm is a direct consequence from the classical preconditioning properties of wavelet base expansions (S. Jaffard [8]).

III.3 General scheme

The numerical algorithm derived from the previous section is now presented in its collocation version. We call I_j the set of points $(\mathbb{Z} \bigcap [0, 2^j[)^2$. Then, $J_j = \{2^{-j}k, k \in I_j\}$ is the two dimensional regular grid of scale 2^{-j} related to $[0, 1[^2$.

For the numerical implementation, we assume that the space $V_p^{\mathcal{P}}$ is such that *any continuous function $f \in V_p^{\mathcal{P}}$ is unambiguously defined by its values on the set of points J_p*. This assumption (satisfied by the even order spline multiresolution we will use in the numerical tests) allows us to define the collocation projection, $C_{V_p^{\mathcal{P}}}$, from the set of real sequences $(f_k)_{k \in I_p}$ to $V_p^{\mathcal{P}}$ as

$$\forall (f_k)_{k \in I_p}, C_{V_p^{\mathcal{P}}}((f_k)_{k \in I_p}) = f \ \Leftrightarrow f \in V_p^{\mathcal{P}} \text{ and } f(2^{-p}k) = f_k, \forall k \in I_p.$$

As soon as we define $f \in V_p^{\mathcal{P}}$ by its coordinates on the basis $\Phi_{pk}^{\mathcal{P}}$, the operator $C_{V_p^{\mathcal{P}}}$ appears as a discrete convolution operator involving a so-called interpolant filter $I_{\Phi_p}(\xi)$. The operator $C_{V_p^{\mathcal{P}}}^{-1}$ is also a convolution operator called the

¶Again, $\Lambda_m^n = \cup_{j=n}^m \Lambda_j$ with, in the periodized framework, $\Lambda_j = \{\lambda = 2^{-j}\left(k + \frac{\varepsilon}{2}\right), k \in \mathbb{Z} \bigcap [0, 2^j[^2, \varepsilon \in E^*\}$.

point value operator and involves the point value filter $PV_{\Phi_p}(\xi)$ defined from $\{\Phi_{p0}(J_p)\}$. Obviously, $I_{\Phi_p}(\xi)$ and $PV_{\Phi_p}(\xi)$ are inverse. Let us remark however, that the point value operator always exists as soon as the functions $\Phi^{\mathcal{P}}_{pk}$ are continuous.

For the implementation, we therefore replace $\Pi_{V_p^{\mathcal{P}}}$ by $C_{V_p^{\mathcal{P}}}$ in (42) and define therefore u_p as

$$u_p = L^{-1} C_{V_p^{\mathcal{P}}} f \ .$$

Given the point values of f on the grid points J_p , the algorithm provides the values of u_p on the same grid. More precisely, the algorithm can be presented as follows:

1. The input of the procedure is the set of values $(f(J_p))$ from which the interpolant function $f_p \in V_p^{\mathcal{P}}$ is constructed using $I_{\Phi_p}(\xi)$:

$$f_p = \sum_{k \in I_p} c_{pk} \Phi^{\mathcal{P}}_{pk}$$

2. f_p is then decomposed into the wavelet subspaces $W_j^{\mathcal{P}}, 0 \leq j \leq p-1$ and $V_0^{\mathcal{P}}$ as

$$f = \sum_{\lambda \in \Lambda_0^{p-1}} \langle f, \Psi^{\mathcal{P}}_{\lambda} \rangle \Psi^{\mathcal{P}}_{\lambda} + c_{00}$$

   where $c_{00} = \Pi_{V_0^{\mathcal{P}}}(f)$.

3. u_p then becomes

$$u_p = \sum_{\lambda \in \Lambda_0^{p-1}} 2^{-js} \langle f, \psi^{\mathcal{P}}_{\lambda} \rangle \widetilde{\theta}^{\mathcal{P}}_{\lambda} + c'_{00}$$

   where $\widetilde{\theta}^{\mathcal{P}}_{\lambda} = \left(2^{js} L^{-1} \psi_{\lambda}\right)^{\mathcal{P}}$. Here, $c'_{00} = c_{00}/\sigma(0)$ for non-homogeneous operators. For homogeneous operator c'_{00} should be given. Note that in that case $\sigma(0) = 0$ and f should have at least s vanishing moments; the fact that c'_{00} should be given corresponds to the ill posed property of the initial problem in L^2.

4. u_p is then expanded in terms of the set of functions $\{\widetilde{\tau}^{\mathcal{P}}_{pk},\ k \in I_p\}$ using the tree algorithms related to $\widetilde{\theta}^{\mathcal{P}}_{\lambda}$ and $\widetilde{\tau}^{\mathcal{P}}_{pk}$ as

$$u_p = \sum_{k \in I_p} c_{pk} \widetilde{\tau}^{\mathcal{P}}_{pk} \ .$$

5. Finally, the grid point values of u_p on J_p are estimated using the point value filter $PV_{\widetilde{\tau}_p}(\xi)$.

It appears clearly that various precalculations should be performed. In the first step, the interpolant filter related to Φ_{pk} must be known; for the second step, the orthogonal multiresolution analysis quadrature mirror filters M_ε should be used and, for the fourth step the corresponding biorthogonal multiresolution quadrature mirror filters $\widetilde{P_\varepsilon}$ are required; finally, the point value filter related to $\widetilde{\tau_p}$ is used for the last step.

To be more precise, we have to make some remarks that help to reduce the complexity and storage. For steps one and two, tensorial properties can be used in a very classical way to reduce the 2D-algorithms involved in $2(\dim(V_p^{\mathcal{P}}))^{1/2} \times$ 1D-algorithms. It is then enough to know the one-dimensional interpolant filter related to $\varphi_{pk}^{\mathcal{P}}$ and the one dimensional quadrature mirror filters m_ξ, $\xi \in E$. For steps four and five, where the filters $\widetilde{P_\varepsilon}$, and the point value filter $PV_{\widetilde{\tau_p}}$ are involved, we can note apply this simplification and we have a full 2D-problem.

We are now able to summarize all these precalculations in the following step 0:

0. The computation of the following filters is performed (this is presented for the spline multiresolution analysis case):

-Interpolant filter related to φ_{pk}, I_{φ_p}: analytical formulas in one dimension are available in V. Perrier and C. Basdevant [15].

-Orthogonal 1D multiresolution analysis filters m_ε: analytical formulas are also available.

-Filters $\widetilde{P_\varepsilon}$ of the Biorthogonal Multi resolution Analysis: These filters are constructed from m_ε and formula (23) and (24). In fact only $\widetilde{P}_{(0,1)}$ and $\widetilde{P}_{(1,1)}$ have to be computed since we have $\widetilde{P}_{(1,0)}(\xi_1,\xi_2) = \widetilde{P}_{(0,1)}(\xi_2,\xi_1)$.

-Point value filter, $PV_{\widetilde{\tau_{pk}}}$ related to $\widetilde{\tau_{pk}}$: This filter is computed from formula (28) and the analytical expression of S, σ and $\hat{\Phi}(\xi)$. We have successively

$$\widetilde{\tau}_p^{\mathcal{P}}(x) = 1/(2\pi) \sum_{w \in \mathbb{Z}^2} \widehat{\widetilde{\tau}}_p(w) e^{2i\pi w.x} \ , \ x \in I\!R^2 \ ,$$

$$\widetilde{\tau}_p^{\mathcal{P}}(x_n) = 1/(2\pi) \sum_{w \in I_p} \left(\sum_{r \in \mathbb{Z}^2} \widehat{\widetilde{\tau}}_p(w + 2^p r) \right) e^{2i\pi w.x_n} \ , \ x_n \in J_p \ . \qquad (43)$$

Practically, 43 is truncated according to a prescribed precision. This is possible because $\hat{\widetilde{\tau}}_p(\xi)$ decreases fast.

Remarks:

- One should again emphasize that the entire algorithm is based on convolution operators. Thanks to the periodic boundary conditions, the convolutions can either be performed directly or using a discrete Fourier

transform. The implementation presented in this paper uses the Fourier transforms since it is optimal for non compactly supported filters on non adapted spaces of approximation.

III.4 Detailed Algorithm

This section is devoted to the structure of the code. Basic tools, such as Fast Fourier Transforms, Convolution/Decimation algorithms, or Term by term multiplications are not described.

As can be seen from the general scheme presented above, the main code involves only two more elaborate routines that will be called the *Precalculus routine* (step 0), and the tree algorithm routines. The tree algorithm routines may or may not use the tensorial structure. They will be called consequently *2D Tensorial Tree Algorithm-D* (steps 2) and *2D Non Tensorial Tree Algorithm-I* (step 4) where -*D* and -I stand for direct and inverse. We recall that the steps 1 and 5 are convolutions and the renormalization performed in step 3 is term by term multiplication.

The tree algorithm routines are becoming very classical and therefore we will not describe them either. Note however, that since only convolutions are performed in our algorithm, we only use the discrete Fourier transform of the wavelet coefficients (and not the corresponding values) at every scale, that reduces significantly the complexity.

We now give the detailed description of the main program (`ELLIP`) and of the precalculus program (`PRECAL`) in pseudo code.

The following example sketches the structure our programs.

```
[OUTPUTS]=Program(INPUTS)
# Comments
# Body of Program:
{
  [OUTPUTS1]= Subprogram1(INPUTS)
  INPUTS2   = OUTPUTS1
  [OUTPUTS] = Subprogram2(INPUTS2)
}
```

Our variable descriptors bears some resemblance to the C language as well as to the *MATLAB* conventions.

III.4.1 Preliminary computations

The symbols $'$, $*$, $.*$, and $./$ used to present this program are borrowed from *MATLAB* and mean respectively, the transposition, the matrix product, the term by term product, and the term by term division. We also use the following n sub sampling operator $a : n : b$ defined as: If a is a 2D array of size $2^p \times 2^p$,

$b(1:2^n:2^p, 1:2^n:2^p)$ is a new array of size $2^{p-n} \times 2^{p-n}$ given by $a(i,j) = a(2^n i, 2Nj)$, $(i,j) \in I_p$.

Program PRECAL

```
[QMFBIW,TAUTW] = PRECAL(p,pmax,QMFW,PHIW,SW,CW,SIGMA)
#INPUT:
#p    -> index of the approximation space Vp in which the
#     elliptic  problem is solve.
#pmax -> index of the approximation space Vpmax in which the
#     precomputation of TAUTILDE is done (it depends on the
#     prescribed precision).
#QMFW ->  structure containing the quadrature mirror  filters in
#     one dimension:
#     QMFW.m0 -> 1D array containing the quadrature mirror filter
#             coefficients associated to the scaling functions;
#             size(QMFW.m0)->2^p; QMFW.m0(i) = m0(i/2^p),
#             i belong to {0,...,2^p-1}.
#     QMFW.m1 -> 1D array containing the quadrature mirror filter
#             coefficients associeted to the wavelet;
#             size(QMFW.m1)->2^p;
#             QMFW.m1(i) = m1(i/2^p),i belong to {0,...,2^p-1}.
#PHIW -> 1D array; size(PHIW)->2^pmax; where pmax is given
#     and pmax>p; PHIW(i) = the value of the Fourier transform
#     of the 1D scaling function at the point i, i belong to
#     {0,...,2^pmax-1}. Used to compute the value of tautilde on
#     the finer grid.
#SW   -> 2D array containing the sampling of the function
#     S used for biorthogonal filters;
#     Size(SW)->(2^pmax X 2^pmax); SW(i,j)=S(i/2^pmax,j/2^pmax),
#     (i,j) belong to {0,...,2^pmax-1}^2.
#
#CW     -> 2D array containing the sampling of the function
#       S(2w)/(2^s S(w)) Size(CW) -> (2^p X 2^p); CW(i,j) =
#        S(2i/2^p,2j/2^p)/ S(i/2^p,j/2^p),
#       (i,j) belong to {0,...,2^p-1}^2.
#SIGMA  -> 2D array containing the sampling of the symbol
#       of the operator Size(SIGMA) -> (2^pmax X 2^pmax);
#       SIGMA(i,j) = sigma(i/2^pmax,j/2^p),
#       (i,j) belong to {0,...,2^pmax-1}^2.
#
#OUTPUT:
#QMFBIW -> structure containing the biorthogonal quadrature
#       mirror filters related to  the tautilde functions:
#          QMFBIW.PTILDE0-> 2D tab containing the biorthogonal
```

```
#                       quadrature mirror filters associated
#                       to the scaling functions;
#                       size(QMFBIW.PTILDE0)->(2^p X 2^p):
#          QMFBIW.PTILDE1-> 2D tab containing the biorthogonal
#                       filters associated to the first wav-
#                       elet;size(QMFBIW.PTILDE1)-> (2^p X 2^p).
#          QMFBIW.PTILDE2-> same as  QMFW.PTILDE1 for the second
#                       wavelet (not computed QMFBIW.PTILDE2 =
#                       QMFW.PTILDE1 transposed).
#          QMFBIW.PTILDE3-> same as QMFW.PTILDE1 for the third
#                       wavelet.
#

#
# Computation of the filters Ptilde
#
QMFBIW.PTILDE0 = ((QMFW.m0)' * (QMFW.m0)) .* CW;
QMFBIW.PTILDE1 = ((QMFW.m0)' * (QMFW.m1)) ./
                 (2^s SW(1:2^(p-pmax):2^pmax,1:2^(p-pmax):2^pmax);
QMFBIW.PTILDE3 = ((QMFW.m1)' * (QMFW.m1)) ./
                 (2^s SW(1:2^(p-pmax):2^pmax,1:2^(p-pmax):2^pmax);
#
# Computation  of the point value filter related to TAUTILDE
#
TAUTW = ( (2^(ps) * PHIW' * PHIW) .* SS)./ SIGMA;
TAUTW = Periodize(TAUTW, p);
```

The subroutine Periodize is not described here, but it is a straight forward transcription of 43.

III.4.2 Main Program

Main Program Ellip

```
[UX] = Ellip(FX,QMFW,FIW,QMFBIW,TAUTW)
#INPUT:
#FX   -> 2D array containing the sampling of the function
#     f; Size(FX) -> (2^p X 2^p); FX(i,j) = f(i/2^p,j/2^p),
#     (i,j) belong to {0,...,2^p-1}^2.
#FIW  -> 1D tab of data containing the interpolation filter
#     related to PHI_p0, size(FIW)->2^p;
#QMFW   -> see PRECAL
#QMFBIW -> see PRECAL
#TAUTW  -> see PRECAL
#
```

```
#
#OUTPUT:
#UX   -> 2D array containing the sampling of the approxi-
#     -mation u_p; size(UX)->(2^p X 2^p), UX(i,j) =
#     u(i/2^p,j/2^p), (i,j) belong to {0,...,2^p-1}^2.
#
#TAUTW  -> 2D array containing the values of the Fourier
#       transform of the scaling function TAUTILDE at level p
#
#TEMPORARY DATA:
#FW   -> 2D tab containing the fft of FX; Size(FW) ->
#     (2^p X 2^p);
#CPW  -> 2D tab containing the fft of scaling coefficient of
#     FX;  Size(CPW) -> (2^p X 2^p);
#DJW  -> Structure of 2D array containing the Fourier transform
#      of the  wavelet coefficients; size(DJW) -> (2^p X 2^p);
#CTILDEPW
#     -> same as CPW for UX;
#UW   -> 2D tab containing the fft of  UX;
#
{
#
# step 0
#
    [FW] = Fast Fourier Transform(FX)
#
# step 1
#
    [CPW] = FW.*FIW (Term by term multiplication)
#
# step 2
#
    [DJW] = 2D Tensorial Tree Algorithm_D (CPW,QMFW)
#
# step 3
#
    [DJW] = DJW.*(2^js) (Term by term multiplication)
#
# step 4
#
    [CTILDEPW] = 2D Non Tensorial Tree Algorithm_I (DJW,QMFBI)
#
# step 5
#
```

```
    [UW] = CTILDEPW.*TAUTW (Term by term multiplication)
#
    [UX] = Inverse Fast Fourier Transform(UW)
}
```

III.5 Storage and Complexity Analysis

As the computation is clearly separated into precalculations and actual implementation of the algorithm, we will also present the storage and complexity analyses separating the two parts. One should remember that the precalculation is done once and for all while, as it will be the case in section IV, the algorithm can be applied iteratively.

We will not discuss the complexity related to one-dimensional computations as well as the storage connected to one-dimensional arrays since both can be neglected in our bidimensional implementation. All the evaluations are performed for $N = \dim V_p = 2^p \times 2^p$.

Storage

Permanant storage (precalculations): The structures `QMFBIW` and `TAUTW` represent four bidimensional arrays of size N.

Temporary storage (actual algorithm): The storage related to bidimensional arrays can be reduced to one arrays of size N.

Finally, the *total memory used* corresponds to five arrays of size N.

Complexity analysis

Precalculus: The computation of the four arrays in the structure `QMFBIW` is done in $C \times N$.

The computation of $PV_{\tau_{p0}}^{\sim_p}$ is performed in $C \times N$ operations. The value of C depends on the precision of the calculation.

Main program:

`Fast Fourier Transform` and `Inverse Fast Fourier Transform` involve $C \times N \log(N)$ operations.

The complexity of the `Term by term multiplication` is N.

`Tree algorithm-D` and `Tree algorithm-I` are based on convolution and decimation operators. These procedures involve therefore $C \times N$ operations.

Therefore the *total complexity* is $O(N \log(N))$.

In the following section we use these programs iteratively,to solve the 2D Burgers equation after reducing it to a cascade of elliptic problems. We would

like to emphasize moreover, that our approach can be also used to solve equations involving homogeneous pseudo-differential operators. A characteristic example is $\sqrt{-\Delta}\,u = f$ with periodic boundary conditions on $[0,1]^2$. We have $L = \sqrt{-\Delta}$ and therefore $\sigma(\xi) = \sqrt{\xi_1^2 + \xi_2^2}$. The most natural choice for S is $S(\xi) = 2\sqrt{\sin^2(\xi_1/2) + \sin^2(\xi_2/2)}$ and one easily checks that the hypotheses of theorem II.2 are then satisfied. The algorithms presented previously can be used (see Pj. Ponenti [16]).

IV Numerical Application: Resolution of the 2D Burgers equations

In this section, we will use the periodized Battle-Lemarié's multiresolution analysis of splines of order m (see P.G. Lemarié [9]). The existence of collocation projectors related to the spline breakpoints requires splines of even order and the value $m = 8$ will be used in the applications.

As described in J. Liandrat et al. [11], any parabolic equation of the type

$$\begin{cases} \frac{\partial u}{\partial t} + L_0 u + G(u) = 0 \\ u(0,t) = u(1,t) \\ u = u_0 \text{ for } t = 0 \\ 0 \leq t \leq Tmax, 0 \leq x \leq 1 \end{cases} \tag{44}$$

where L_0 is a differential operator of even order with positive symbol $\sigma_0(\omega)$, and G is generally a nonlinear function of u and its derivatives, can be numerically approached using a classical finite difference time discretization scheme followed by a variational approximation of the resulting elliptic problems. We show now that the approach developed in the previous sections can be used efficiently to provide this approximation.

Following J. Liandrat et al. [11] we first introduce a segmentation $\{t_n\}_{n=1}^M$ of $[0, Tmax]$ (i.e. a sequence $\{t_n\}_{n=1}^M$ such that $0 = t_0 < t_1 < ... < t_M = Tmax$) and now look for a sequence of functions of the x variable $\{u^{(n)}(x)\}_{n=1}^M$ such that $u^{(n)}(x)$ is an approximation of $u(x, t_n)$.

With $\Delta t_n = t_{n+1} - t_n, 0 \leq n < M$, and considering first (44) as an ordinary differential equation in time, a standard finite-difference discretization leads to the following iterative equation:

$$\mathcal{L}_n u^{(n+1)} = F(u^{(n)}, ..., u^{(n-i)}, \Delta t_n, ..., \Delta t_{n-i}, G(u^{(n)}), ..., G(u^{(n-j)})) \tag{45}$$

where $\mathcal{L}_n$ is a step forward operator that together with F is determined by the choice of the finite-difference approximation of the time-dependent ordinary

differential equation. We always assume that this approximation is at least semi implicit for the linear part L_0 and explicit for the nonlinear part. Therefore $\exists \alpha > 0$ such that $\mathcal{L}_n = (I + \alpha \Delta t_n L_0)$ where I stands for the identity operator. By hypothesis, $\sigma_0(\omega) \geq 0, \forall \omega$ and then $\mathcal{L}_n$ has always a symbol bounded from below by 1.

Hence, assuming that $\{u^{(n-l)}, l = 0, ..., i\}$ and $\{G(u^{(n-l)}, l = 0, ...j\}$ are known, the resolution of (45) falls under the scope of paragraph III.3 and the resolution of (44) can be therefore performed iteratively.

The bidimensional Burgers equation writes, with $u = (u_1, u_2)$:

$$\begin{cases} \frac{\partial u}{\partial t} + \nabla u.u = \nu \Delta u \\ u(0,t) = u(1,t) \\ \\ u = u_0 \text{ for } t = 0 \\ \\ 0 \leq t \leq Tmax, 0 \leq x \leq 1. \end{cases} \tag{46}$$

Choosing a constant step segmentation of $[0, Tmax]$ (i.e., a segmentation such that $\exists \Delta t$ such that $\forall 0 \leq n \leq M, t_n = n\Delta t$), an implicit Crank-Nicholson time scheme (second order) for the linear term $(\nu \Delta u)$, and an explicit second order Adams-Bashforth scheme for the nonlinear term $(\nabla u.u)$ we get

$$\left(I - \nu \frac{\delta_t}{2} \Delta\right) u^{(n+1)} = \left(I + \nu \frac{\delta_t}{2} \Delta\right) u^{(n)} - \delta_t (\frac{3}{2} \nabla u^{(n)}.u^{(n)} - \frac{1}{2} \nabla u^{(n-1)}.u^{(n-1)})$$

and the solution can be written as

$$u^{(n+1)} = \tilde{u}^{(n+1)} - u^{(n)} ,$$

with

$$\tilde{u}^{(n+1)} = \left(I - \nu \frac{\delta_t}{2} \Delta\right)^{-1} \left(2u^{(n)} - \delta_t (\frac{3}{2} \nabla u^{(n)}.u^{(n)} - \frac{1}{2} \nabla u^{(n-1)}.u^{(n-1)})\right) \tag{47}$$

To fall completely under the scope of paragraph III.3, one should be able to evaluate the point values of the nonlinear term of (47). We used the simplest method available that consists, as classically done in spectral methods (C. Canuto et al. [3]), to "apply the nonlinear operator on the grid points". More precisely, the approximation of $\nabla u^{(l)}.u^{(l)}$ we used is $PV(\nabla u^{(l)}.u^{(l)}) = C_{V_p^{\mathcal{P}}}^{-1}(C_{V_p^{\mathcal{P}}} \times (\nabla u^l).C_{V_p^{\mathcal{P}}}(u^l))$ where $\times$ is a term by term multiplication of finite sequences and $C_{V_p^{\mathcal{P}}}$ is the collocation projection introduced in section III.2.

Then, for each time step $n\Delta t$, the problem clearly belongs to the class of elliptic problems studied in section (III) with $L = I - \nu \frac{\delta_t}{2} \Delta$ and $f = 2u^{(n)} - \delta_t (\frac{3}{2} \nabla u^{(n)}.u^{(n)} - \frac{1}{2} \nabla u^{(n-1)}.u^{(n-1)})$. The iterative form of the equation induces some modifications of the general scheme presented in III.3 and we therefore provide the full scheme for an iteration of the Burgers approximation scheme:

0. The inputs are the values of $(u^{(n)}(J_p))$, $(\frac{\partial u^{(n)}}{\partial x_1}(J_p))$ and $(\frac{\partial u^{(n)}}{\partial x_2}(J_p))$.

1. $F^{(n)}(J_p) = 2u^{(n)}(J_p) - \delta_t PV(\nabla u^{(n)}.u^{(n)})(J_p)$ is computed as described above.

2. $F_p^n(x)$, the function of $V_p^{\mathcal{P}}$ interpolating the values $F^n(J_p)$ is constructed as $F_p^n = \sum_{k\in I_p} c_{pk}\Phi_{pk}^{\mathcal{P}}$,

3. F_p^n is then decomposed into the wavelet subspaces $W_j^{\mathcal{P}}, 0 \le j \le p-1$ and $V_0^{\mathcal{P}}$ as

$$F_p^n = \sum_{\lambda\in\Lambda_0^{p-1}} \langle F_p^n, \Psi_\lambda^{\mathcal{P}}\rangle \Psi_\lambda^{\mathcal{P}} + c_{00}$$

where $c_{00} = \Pi_{V_0^{\mathcal{P}}}(F_p^n)$.

4. $u_p^{(n+1)}$ then becomes

$$u_p^{(n+1)} = \sum_{\lambda\in\Lambda_0^{p-1}} 2^{-js}\langle F_p^n, \psi_\lambda^{\mathcal{P}}\rangle \widetilde{\theta}_\lambda^{\mathcal{P}} + c_{00}$$

where $\widetilde{\theta}_\lambda^{\mathcal{P}} = \left(2^{js}L^{-1}\psi_\lambda\right)^{\mathcal{P}}$.

5. $u_p^{(n+1)}$ is then expanded in terms of the set of functions $\{\widetilde{\tau}_{pk}^{\mathcal{P}},\ k \in I_p\}$ as

$$u_p^{(n+1)} = \sum_{k\in I_p} c_{pk}\widetilde{\tau}_{pk}^{\mathcal{P}} .$$

We also get $\frac{\partial}{\partial x}\tilde{u}^{(n+1)} = \sum_{k\in I_p} c_{pk}\frac{\partial}{\partial x}\widetilde{\tau}_{pk}^{\mathcal{P}}$
and $\frac{\partial}{\partial y}\tilde{u}^{(n+1)} = \sum_{k\in I_p} c_{pk}\frac{\partial}{\partial y}\widetilde{\tau}_{pk}^{\mathcal{P}}$.

6. From the point values of $\widetilde{\tau}_p^{\mathcal{P}}(J_p)$, $\frac{\partial}{\partial x}\widetilde{\tau}_p^{\mathcal{P}}(J_p)$, and $\frac{\partial}{\partial y}\widetilde{\tau}_p^{\mathcal{P}}(J_p)$ we compute $(\tilde{u}^{(n+1)}(J_p))$ and its gradient using the corresponding point value filters.

7. Finally using the values of $(u_n(J_p))$ and of its gradient, we get the values of $(u^{(n+1)}(J_p)) = (\tilde{u}^{(n+1)}(J_p) - u^{(n)}(J_p))$ and its gradient.

IV.1 Storage and Complexity Analysis

As described above, the numerical code implements the elliptic solver in an iterative process. Since the time step Δt is constant, the characteristics of the elliptic solver does not depend on the time index n. Then, the solver precalculations related to the whole parabolic problem are the same as the ones related to the elementary elliptic solver (see section III.5). This applies to the storage and to the complexity as well.

As it has been shown in the previous section however, extra work, not connected to the elliptic solver itself but to the computation of the right hand side term of the iterative equation (47) is required. This extra work is related to the storage of the fields at the different time steps involved in the three level time step Adams-Bashforth Crank-Nicholson scheme and to the point value evaluation of the derivatives involved in the nonlinear part. Again, it can be split into permanent and temporary storage as well as in precalculation and main program extra work.

Storage (in addition to the elliptic solver storage)

Permanant storage (precalculations): One extra structure containing the point values $\frac{\partial}{\partial x}\widetilde{\tau}_p^{\mathcal{P}}(J_p)$ must be stored in one bidimensional array of size N; the structure containing the point values $\frac{\partial}{\partial y}\widetilde{\tau}_p^{\mathcal{P}}(J_p)$ is given by transposition of the previous one.

Temporary storage (actual algorithm): The two fields $u^{(n)}$, $u^{(n-1)}$ and $u^{(n+1)}$ can be handled using three arrays of size N.

Complexity analysis (in addition to the elliptic solver complexity)

Precalculation: The computation of $PV_{\frac{\partial}{\partial x}\widetilde{\tau}_p^{\mathcal{P}}}$ is performed in $C \times N$ operations where, as in section III.5 the value of C depends on the precision of the calculation.

Main program: The addition of complexity is related to steps (1), (5), (6) and (7). Since these steps involve convolutions and term by term products, the added complexity is again $CN \log N$

Finally, the *total memory used* is 7 arrays of size N.

The *total complexity* is $O(N \log(N))$.

Obviously, the total complexity of the whole resolution is M times the complexity of one time step resolution.

IV.2 Numerical Results

Test case on an x_2 translation invariant problem

The validation of our code has been performed on an x_2 translation invariant problem constructed using for the initial condition $(u_{0_1}, u_{0_2})(x_1, x_2) = (\sin(2\pi x_1), 0)$. Indeed, with such an initial condition, the solution remains x_2 translation invariant.

For an easy comparison to the well documented paper of C. Basdevant et al. [1] we used $\nu = 10^{-2}/\pi$.

As explained in C. Basdevant et al. [1], the pertinent quantities are

$$ms = sup_t || \frac{\partial u}{\partial x_1}(x,t) ||_\infty = sup_{t \in [0, Tmax]} | \frac{\partial u}{\partial x_1}(0.5, t)|$$

and t_{ms} defined such that

$$|\frac{\partial u}{\partial x_1}(0.5, t_{ms})| = ms.$$

Table 48 exhibits the numerical results obtain using various values for the time step Δt. The maximum time step numerically acceptable was $\Delta t = 0.0075$. In each case, the values of ms are computed by interpolation and the corresponding values of t_{ms} are deduced. The comparison with the expected theoretical values (first column) shows that our method competes favorably with the majority of the schemes presented in C. Basdevant et al. [1]. A complete study of the time step size dependence of the results connected to the stability analysis of the parabolic algorithm will be published later.

Exact	Δ_t	0.0005	0.001	0.0025	0.005
−304.0103	ms	−304.6308	−305.727	−309.4354	−316.5454
0.255237	t_{ms}	0.253	0.252	0.25	0.245

(48)

Test case on a first diagonal translation invariant problem
Our second test case is performed on a first diagonal translation invariant problem constructed using $(u_{0_1}, u_{0_2})(x_1, x_2) = (\sin(2\pi(x_1 + x_2), \sin(2\pi(x_1 + x_2))$. Again, the solution can be compared to the reference solution of C. Basdevant et al. [1] thanks to a 45^o rotation and to a time dilation of factor 2. However, according to our reference axes, it is obviously a fully bidimensional solution.

Figures 1,2 and 3 show the isoline values of the numerical approximations computed with $\delta t = .001$ at $t = 0$, $t = 0.15$ and $t = 0.50$. The first diagonal translation invariance is kept and we obtain the values $ms = 249.0528$ and $t_{ms} = 0.123$. The expected theoretical values are -304.0103 for ms and 0.1276185 for t_{ms}. This is not as good as before but one should note that the resolution in the direction perpendicular to the front axis is now half the one in our previous calculations.

Since the ultimate application of all this work is the development of adaptive algorithms (i.e. the development of algorithms handling approximation spaces of reduced dimension adapted to the solution regularity (see for instance Pj. Ponenti [16]),we have estimated, at various times, the wavelet basis adapted to the approximation and defined as the lowest cardinal $m = 8$ spline wavelet basis preserving the L^2 norm of the approximation with a precision of 10^{-6}. The columns of table (49) show for each scale $0 \leq j \leq 7$ the number of wavelets selected in the adapted basis related to the approximated solution at various times. It appears that, compared to the full basis of V_8 (last column), these bases have a drastically reduced cardinal (we defined the rate of compression as $\frac{\text{cardinal of the adapted basis}}{\text{cardinal of} V_8}$) even if the gradients of the solution fill up a large domain made of two complete lines of the plane (see figures 1, 2 and 3)

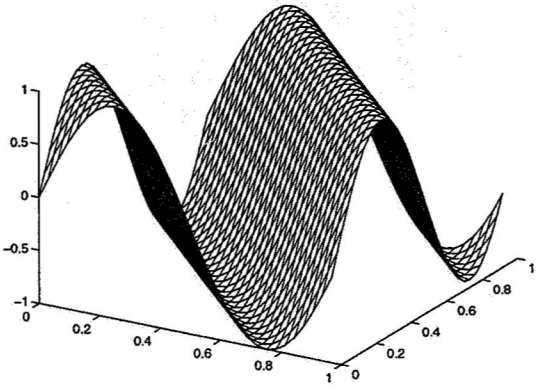

Figure 1: Initial condition, $t_i = 0$.

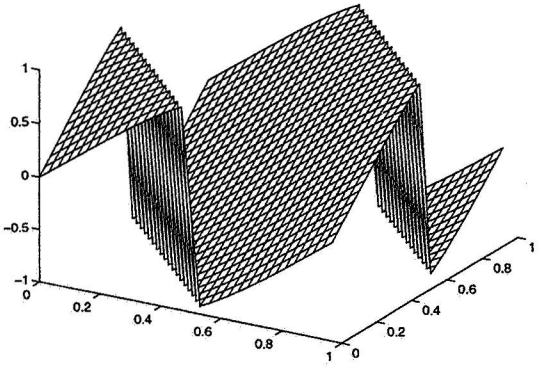

Figure 2: Approximated solution, $t_i = 0.15$.

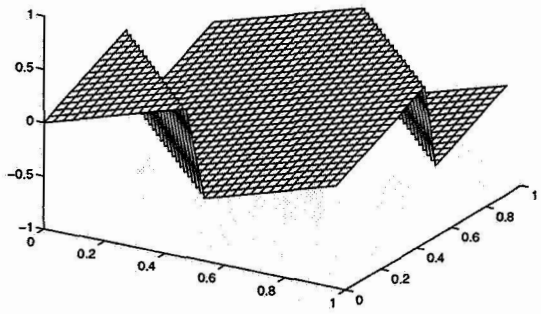

Figure 3: Approximated solution, $t_i = 0.50$.

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Conclusion

In this paper, we have proposed an inversion scheme for elliptic problems based on biorthogonal wavelets. The approximation of elliptic problem solutions is constructed and leads to stable and fast numerical algorithms.

Numerical tests related to the approximation of the parabolic Burgers equations transformed into a cascade of elliptic problems are provided.

The approximation scheme is based on convolution operators and can therefore be theoretically used in the framework of adapted spaces of approximation. As mentioned however, the nice tensorial product structure that enforces numerical efficiency is fragile and is generally destroyed when applying the scheme directly. Other approximations for the step forward operator, should allow one to use efficiently this approximation in a general context of adapted multidimension spaces of approximation.

	$t_i = 0$	$t_i = 50$	$t_i = 100$	$t_i = 150$	$t_i = 200$	$t_i = 250$	$t_i = 300$	$t_i = 500$	Number of wavelets in W_j
$j = 0$	0	0	0	0	0	0	0	0	3
$j = 1$	8	8	8	8	8	8	8	8	12
$j = 2$	32	40	40	40	40	40	40	40	48
$j = 3$	0	176	176	176	176	176	176	176	192
$j = 4$	0	736	736	736	736	736	736	736	768
$j = 5$	0	2240	3008	3008	3008	3008	3008	3008	3072
$j = 6$	0	0	11008	11904	11904	11648	11264	10624	12288
$j = 7$	0	0	18944	20480	19968	19200	18944	15872	49152
Cardinal of the adapted basis	40	3200	33920	36352	35840	34816	34176	30464	65536
Rate of compression	0.0610	5.0000	51.7578	55.4688	54.6875	53.1250	52.1484	46.4844	

(49)

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Pj. Ponenti
J. Liandrat

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Wavelet bases are used to generate spaces of approximation for the resolution of bidimensional elliptic and parabolic problems. Under some specific hypotheses relating the properties of the wavelets to the order of the involved operators, it is shown that an approximate solution can be built. This approximation is then stable and converges towards the exact solution. It is designed such that fast algorithms involving biorthogonal multi resolution analyses can be used to resolve the corresponding numerical problems.

Detailed algorithms are provided as well as the results of numerical tests on partial differential equations defined on the bidimensional torus.

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